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# Analysis of Surface Wave Characters in the Prediction of Wave Restoring Forces

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**Abstract**
Floating and offshore structures are prone to water waves and are affected by waves in different ways. These structures generate their own waves when oscillating in water. The generated waves have varied impacts on the structures and on other offshore bodies. The commonly known impact is the creation of periodic loads which are the added mass, damping and the restoring moments and forces. Restoring forces act on the body to bring it back to the steady equilibrium state. Restoring forces are therefore important to ocean and marine engineers in modelling of both offshore and floating bodies. To date, very little has been done on this area. In fact, nobody has come up with a connectivity between the outgoing wave characters and the restoring forces. This work is concerned about the outgoing wave characters in the production of corresponding restoring forces. To do this, it is scientifically correct to say that the incoming and outgoing waves have the same characters and due to this similarity, the wave characters of incoming waves have the same characters as those of outgoing. Such to be analyzed include; wave elevation, velocity and vertical acceleration. The aim of this paper, therefore, is to analyze these wave characters and determine how they behave at different heights away from the shore. It is of great importance to note that the analysis of these wave characters was a success because of the presence of a velocity potential which is derived by the method of separation of variables and by use of governing boundary conditions. This paper also touches on the dispersion relation of water waves and an analysis was done to show how these waves disperse at different depths on or away from the shore.

**Index terms:** Dispersion relation, incident waves, restoring forces, outgoing waves, wave elevation, wave velocity, wave acceleration.

## 1. Introduction

Waves have enormous and varied impacts on offshore structures. The effect that a wave causes on offshore structures varies according to the strength of the waves and how these structures are modeled in order to survive strong waves (Abdel Raheem, 2016). Wave also causes motion to offshore structures and ships. The motions caused by the waves reduces the general workability of these structures. They even reduce their effectiveness. With these, there has been considerable attention from designers since they now acknowledge the importance of designing structure with considerations of hydrodynamics loads acting on them (Manyanga and Duan, 2012). Surface ocean waves cause hydrodynamic loads on offshore man-made structures irrespective of whether they are located deep or on the surface of the ocean (Finnegan *et al.,* 2013). Hydrodynamic loads are varied and in order to have a deep understanding of them, it is important to have a good understanding of the physics of water waves (Ngina *et al.,* 2015). Therefore, performing analysis of the incoming waves is of great importance to ocean engineers. These water waves have varied characteristics which need to be looked into separately. Among these characteristics are the incident velocity potential, wave elevation, wave velocity and wave acceleration. The characteristics aforementioned are looked into individually in this study. The solution of the incident velocity potential is arrived at by applying the separation of variables method to the Laplace equation with considerations of governing equations. With the velocity potential, the analysis of wave velocity is done. The influence of the depth of the water is done to the wave elevation, velocity and acceleration and results run at different depths. Since the wave characters of incident waves are the same as the wave characters of outgoing waves (Tam and Webb, 1993), the results from this research serves as a prediction of outgoing wave characteristics.

## 2. Mathematical Formulation and Governing Equations

The fluid motion under study is three dimensional in the $o-xyz$ plane and the waves are propagated in the $x$ direction. The $z$ axis is the vertical axis that is measured upward, opposing the force of gravity, from the undisturbed free surface. The $o-xy$ plane then lies on the undisturbed free surface. The free surface water displacement from its mean surface is given by $z = \eta(x, y, t)$. The free surface water displacement is taken to be the wave elevation. The depth of the water is taken to be the distance between the sea bed ($z = -h$) and the still water level $z = 0$. The wavelength of the wave $\lambda$ is assumed to be greater than the wave amplitude. Since the wave under study is periodic in nature, the wave period is taken as the time used by one wave to pass a particular point. The solution of this problem is governed by specific boundary conditions.

The fluid under study is incompressible and the divergence of its velocity filed is equal to zero and is expressed as;

$$\vec{\nabla}.\vec{V} = 0 \tag{1}$$

The velocity components of the flow in the $x, y, z$ plane is expressed as;

$$u = \frac{\partial \phi}{\partial x}, \mathrm{v} = \frac{\partial \phi}{\partial y}, \mathrm{w} = \frac{\partial \phi}{\partial z} \tag{2}$$

Equation (2) is true due to the fact that the flow under study is irrotational and a velocity potential $\phi(x, y, z)$ exists.

Equation (1) is substituted to equation (2) such that the Laplace equation in equation (3) is satisfied by the velocity potential.

$$\nabla^2 \phi = 0 \tag{3}$$

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$$\frac{\partial^2\phi}{\partial u^2}+\frac{\partial^2\phi}{\partial v^2}+\frac{\partial^2\phi}{\partial w^2}=0 \qquad (4)$$

Boundary conditions are to be satisfied both at the free surface and at the bottom. Therefore, the boundary conditions for this case are as follows;

The velocity components at the seabed, $z=-h$, must go to zero;

$$w=\frac{\partial\phi}{\partial z}=0,\ z=-h \qquad (5)$$

There exists a dynamic condition such that the pressure below the free surface is always equal to the atmospheric pressure with surface tension neglected.

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\left|\nabla\phi\right|^2+\frac{p}{\rho}+gz=0 \qquad (6)$$

From the equation above, the non-linear terms $|\nabla\phi|^2$ are to be neglected for small amplitude waves and thus equation (6) reduces to;

$$\left(\frac{\partial\phi}{\partial t}\right)_{z=0}+g\eta=0 \qquad (7)$$

Where; $z=\eta(x,y,t)$ (8)

Equation (7) is the Dynamic Free Surface Boundary Condition.

Consider a function $F(x,y,z)$ whose material derivative can be expressed in the form;

$$\frac{DF}{Dt}=\left(\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}-z\right)_{z=0}=0 \qquad (9)$$

$$\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)_{z=\eta}+\frac{\partial\eta}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)_{z=\eta}=\left(\frac{\partial\phi}{\partial z}\right)_{z=\eta} \qquad (10)$$

Since, from equation (10), $\left(\frac{\partial\eta}{\partial x}\right)\left(\frac{\partial\phi}{\partial x}\right)_{z=\eta}$ and $\left(\frac{\partial\eta}{\partial y}\right)\left(\frac{\partial\phi}{\partial y}\right)_{z=\eta}$ are small for small-amplitude waves. Equation (10) is reduced to;

$$\frac{\partial\eta}{\partial t}=\left(\frac{\partial\phi}{\partial z}\right)_{z=0} \qquad (11)$$

From equation (7) and equation (11), expression (12) is arrived at as;

$$\frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\phi}{\partial z}=0 \quad \text{at} \quad z=0 \qquad (12)$$

At a time when the velocity potential $\phi$ oscillates harmonically in time with a circular frequency $\omega$, then equation (12) can be expressed as;

$$-w^2\phi+g\frac{\partial\phi}{\partial z}=0 \quad \text{at} \quad z=0 \qquad (13)$$

In these conditions, $g$ is the acceleration due to gravity.

### 3. Velocity Potential

Assuming that the free surface wave elevation is represented in the form below;

$$\eta(x,y,t)=qe^{i(k_1x\cos\theta+k_2y\sin\theta-\omega t)} \qquad (14)$$

The prime aim is finding the velocity potential ($\phi$) that satisfies all the boundary conditions that were mentioned above.

The Laplace equation is solved, in this regard, using the separation of variables method.

Let the incident wave potential take the form given below;

$$\phi=f(z)e^{i(k_1\cos\theta+k_2\sin\theta-\omega t)} \qquad (15)$$

Where $f$ is a representation of the restoring forces.

With differentiation, it follows that;

$$\nabla^2\phi=\frac{\partial^2 f}{\partial z^2}-k^2 f(z)=0 \qquad (16)$$

Where,

$$k^2 = \sqrt{k_1^2 + k_2^2} \quad (17)$$

The solution of (16) is given as;

$$z = Ne^{kz} + Je^{-kz} \quad (18)$$

Putting (18) into (15) gives;

$$\phi = (\mathrm{N}e^{kz} + Je^{-kz})(e^{i(k_1 x\cos\theta + k_2 y\sin\theta - \omega t)}) \quad (19)$$

From equation (4) it follows that;

$$Ne^{-kh} - Je^{kh} = 0$$

$$Ne^{-kh} = Je^{kh} \quad (20)$$

Substituting equation (19) into equation (13) gives;

$$(\omega^2 - gk)\,\mathrm{N} + (\omega^2 + gk)\,\mathrm{J} = 0 \quad (21)$$

Writing equations (20) and (21) in matrix form gives;

$$\begin{vmatrix} e^{-kh} & -e^{kh} \\ (\omega^2 - gk) & (\omega^2 + gk) \end{vmatrix} = 0 \quad (22)$$

$$\omega^2 = gk\left(\frac{e^{kh} - e^{-kh}}{e^{kh} + e^{-kh}}\right) = gk\tanh kh \quad (23)$$

Equation (23) is the dispersion relation equation that equates the wave frequency and celerity in relation to the wave number $k$. Wave celerity is expressed as;

$$c = \frac{\omega}{k} \quad (24)$$

and frequency as;

$$\omega^2 = gk\tanh kh \quad (25)$$

Then;

$$\omega^2 \frac{h}{g} = kh\tanh kh \quad (26)$$

$$\omega\sqrt{\frac{h}{g}} = \sqrt{kh\tanh kh} \quad (27)$$

Letting $Ne^{-kh} = Je^{kh} = \frac{H}{2}$ (28)

Hence equation (19) becomes;

$$\phi = \frac{1}{2}H\left\{e^{k(h+z)} + e^{-k(h+z)}\right\}\left(e^{i(kx\cos\theta + ky\sin\theta - \omega t)}\right) \quad (29)$$

$$\phi = \left(\mathrm{H}\cosh k(h+z)\right)e^{i(kx\cos\theta + ky\sin\theta - \omega t)} \quad (30)$$

But $\eta = \left(\frac{\partial\phi}{\partial t}\right)$ (31)

$$\eta = \frac{i\omega}{g}\mathrm{H}\cosh kh(e^{i(kx\cos\theta + ky\sin\theta - \omega t)} \quad (32)$$

$$\eta = qe^{i(kx\cos\theta + ky\sin\theta - \omega t)} \quad (33)$$

Equation (25) can still be expressed as;

$$\frac{\omega^2}{g}\cosh kh = k\sinh kh \quad (34)$$

$$H = -i\frac{q\omega}{k}\frac{1}{\sinh kh} \quad (35)$$

Therefore;

$$\phi = -i\frac{q\omega}{k}\frac{\cosh k(\mathrm{h}+\mathrm{y})}{\sinh kh}e^{i(kx\cos\theta + kz\sin\theta - \omega t)} \quad (36)$$

However, from Euler's formular for ordinary trigonometric function, equation (36) can be expressed as;

$$\phi = \left(-i\frac{q\omega}{k}\frac{\cosh k(h+z)}{\sinh kh}\right)\left[\mathrm{isin}(\mathrm{kxcos}\,\theta + \mathrm{kysin}\,\theta - \omega t)\right] \quad (37)$$

$$\mathrm{Re}(\phi) = \frac{q\omega}{k}\frac{\cosh k(h+z)}{\sinh kh}\sin\left(kx\cos\theta + ky\sin\theta - \omega t\right) \quad (38)$$

Equation (38) is the velocity potential satisfying the boundary conditions mentioned earlier.

## 4. Wave Characters

Wave elevation is an important wave character since it is used to determine the upward and downward movements of water structures from the undisturbed free surface. This wave character is important to coastal engineers in the prediction of damages that may occur to ships due to slamming (Ngina *et al.,* 2014).

The incident wave elevation is gotten from differentiating equation (38) with respect to time and applying equation (31). That is;

$$\frac{\partial\phi}{\partial t} = \eta = \frac{q\omega^2\cosh k(h+z)}{k\sinh kh}\cos(kx\cos\theta + ky\sin\theta - \omega t) \quad (39)$$

On the other hand, the wave velocity is derived from differentiating (38) partially along each axis. That is;

$$u = \frac{\partial\phi}{\partial x} = \frac{q\omega\cos\theta\cosh k(h+z)}{\sinh kh}\cos(kx\cos\theta + ky\sin\theta - \omega t)$$

$$v = \frac{\partial\phi}{\partial y} = \frac{q\omega\sin\theta\cosh k(h+z)}{\sinh kh}\cos(kx\cos\theta + ky\sin\theta - \omega t) \quad (40)$$

$$w = \frac{\partial\phi}{\partial z} = \frac{q\omega\sinh k(h+z)}{\sinh kh}\sin(kx\cos\theta + ky\sin\theta - \omega t)$$

The equations above express the local velocities of the fluid at a distance $(z+h)$ above the impermeable bottom. The fluid velocities are periodic in nature. The hyperbolic functions in equation (40) are a representation of exponential decays of the magnitude of velocity components in respect to increase of distance below the free surface.

Finally, wave acceleration is derived by differentiating (40) with respect to time. That is;

$$a_1 = u_t = \frac{q\omega^2\cos\theta\cosh k(h+z)}{\sinh kh}\sin(kx\cos\theta + ky\sin\theta - \omega t)$$

$$a_2 = v_t = \frac{q\omega^2\sin\theta\cosh k(h+z)}{\sinh kh}\sin(kx\cos\theta + ky\sin\theta - \omega t) \quad (41)$$

$$a_3 = w_t = -\frac{q\omega^2\sinh k(h+z)}{\sinh kh}\cos(kx\cos\theta + ky\sin\theta - \omega t)$$

The negative sign shows that the acceleration of the wave is changing with respect to the origin.

## 5. Results and Discussion

As a way of getting proper understanding of wave restoring forces and moments, it is important to have a clear understanding of how wave characters change with respect to the distance away from the shore. Since wave characters of incoming and outgoing waves are quite the same, the characteristics of incident waves was used in this research to predict the characteristics of outgoing waves.

In figure 1, the $y-$ $axis$ shows the wave angular frequencies and the $x-$ $axis$ on the other hand shows the wave number $k$. From the graph it is evident that the waves disperse differently at different heights away from the shores. It is observed that greater wavelengths are observed when the waves are identified far from the shore at large heights. With the data from the graph, it is quite easier making decisions on where to construct offshore structures in the oceans or the seas. It will then be of great security and advantage to construct offshore structures at distances near the shoreline where little or no wave dispersion is experienced (Masria *et al.,* 2015). In scenarios where offshore structures are constructed away from the shore; the chances are that they will experience adverse effects of the waves and they may get washed away or destroyed over time.

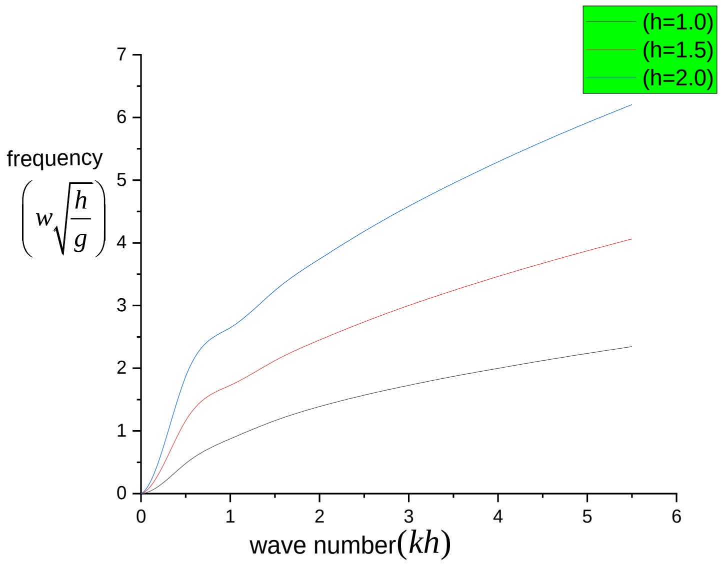


Figure 1: Graph showing the dispersion relation of incident waves at different water depths.

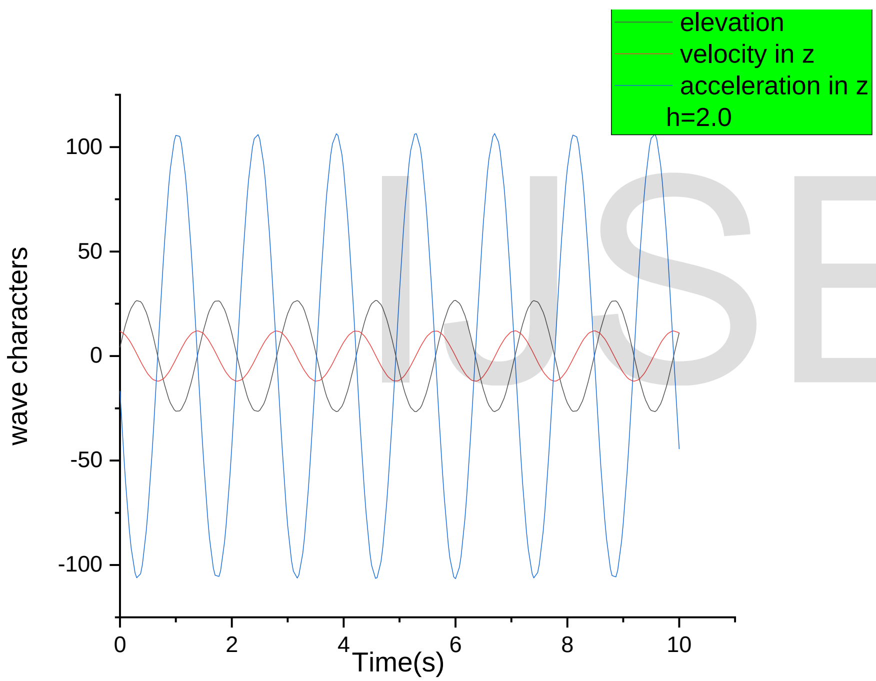


Figure 2: Graph showing the relationship between wave elevation, velocity and acceleration in z direction for water depth h=2.0

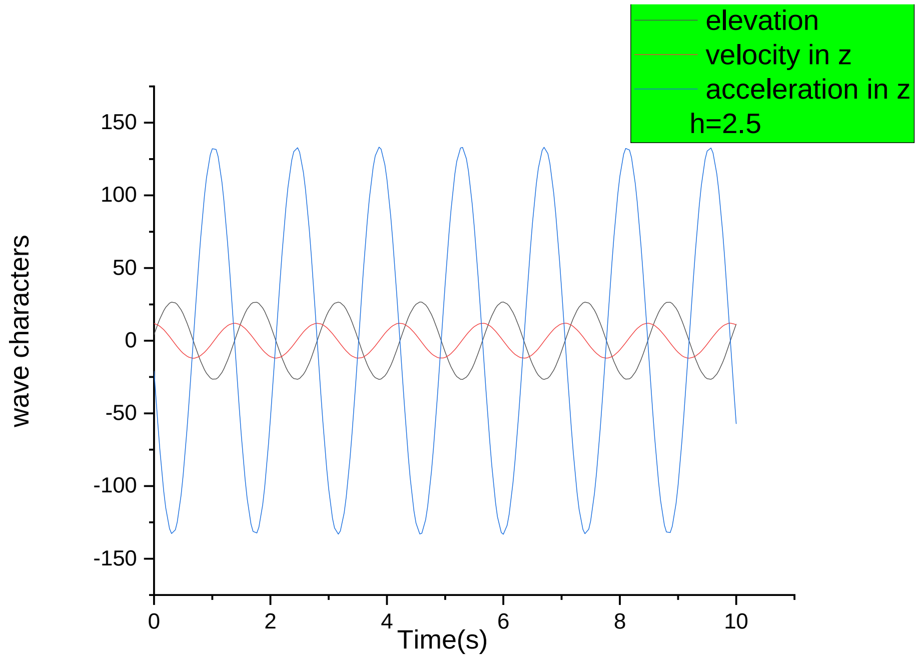


Figure 3: Graph showing the relationship between wave elevation, velocity and acceleration in z direction for water depth h=2.5

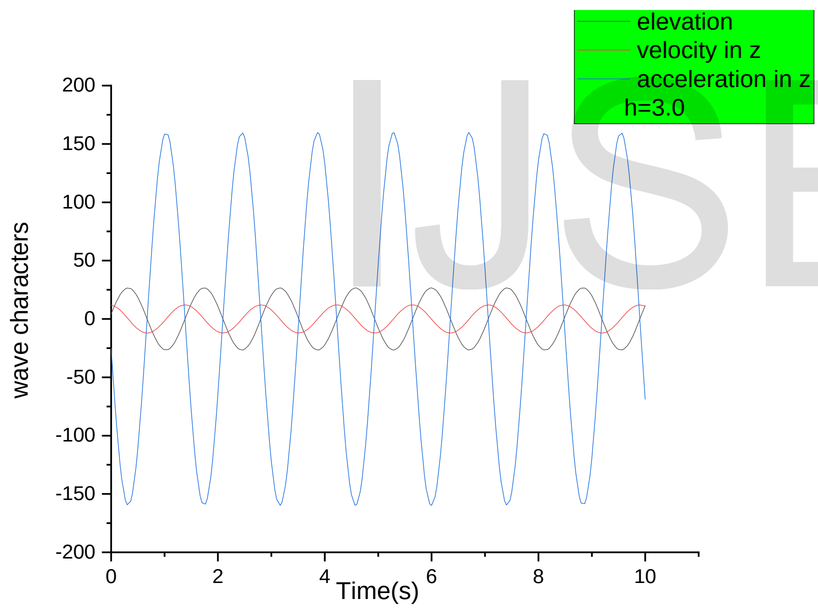


Figure 4: Graph showing the relationship between wave elevation, velocity and acceleration in z direction for water depth h=3.0

Figures 2, 3, 4 are a representation of the relationships between wave elevation, velocity and acceleration on depths near and away from the shores. All this wave characters were examined with respect to the vertical direction.

It is evident, from figures 2, 3 and 4, that with changes in depth the wave accelerations keep changing while elevation and velocity remains constant. From these figures it is worth noting that velocity and displacement are 90˚ out of phase while displacement and acceleration are 180˚ out of phase (Brain *et al.,* 2015). It is also good to note that the amplitudes of wave acceleration are quite higher than those of wave velocity. The changes in wave acceleration that is depicted in these graphs are because of the changes in water frequencies with distances away from the shore. As the distance away from the shore increases, the wave frequencies increase and so does the wave amplitudes and this results in high accelerations.

**Conclusion**

The wave characters in relation to wave restoring forces were derived. It was shown that the waves disperse more towards the deeper part of the sea than at the shallow part. On the other hand, the acceleration of the waves was observed to be increasing with an increase in depth and this is due to the fact that the wave frequencies increase with increase in depth and so does the amplitude. Hence, it is true to conclude from the above predictions that restoring forces of water waves increases with depth. So, the engineering practices in the ocean or the sea should consider these facts in construction and under water activities.

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